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THE O(N) VECTOR MODEL IN THE LARGE N LIMIT REVISITED: MULTICRITICAL POINTS AND DOUBLE SCALING LIMIT

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ABSTRACT

The multicritical points of the $O(N)$ invariant N vector model in the large N limit are reexamined. Of particular interest are the subtleties involved in the stability of the phase structure at critical dimensions. In the limit $N \to \infty$ while the coupling $g \to g_c$ in a correlated manner (the double scaling limit) a massless bound state $O(N)$ singlet is formed and powers of $1/N$ are compensated by IR singularities. The persistence of the $N \to \infty$ results beyond the leading order is then studied with particular interest in the possible existence of a phase with propagating small mass vector fields and a massless singlet bound state. We point out that under certain conditions the double scaled theory of the singlet field is non-interacting in critical dimensions.

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1 Introduction

Statistical mechanical properties of random surfaces as well as randomly branched polymers can be analyzed within the framework of large N expansion. In the same manner in which matrix models in their double scaling limit [1,2,3,4] provide representations of dynamically triangulated random surfaces summed on different topologies, $O(N)$ symmetric vector models represent discretized branched polymers in this limit, where $N \to \infty$ and the coupling constant $g \to g_c$ in a correlated manner [5,6]. The surfaces in the case of matrix models, and the randomly branched polymers in the case of vector models are classified by the different topologies of their Feynman graphs and thus by powers of 1/N. Though matrix theories attract most attention, a detailed understanding of these theories exists only for dimensions $d \leq 1$. On the other hand, in many cases, the $O(N)$ vector models can be successfully studied also in dimensions $d > 1$ [6], and thus, provide us with intuition for the search for a possible description of quantum field theory in terms of extended objects in four dimensions, which is a long lasting problem in elementary particle theory.

The double scaling limit in $O(N)$ vector quantum field theories reveals an interesting phase structure beyond $N \to \infty$ limit. In particular, though the $N \to \infty$ multicritical structure of these models is generally well understood, there are certain cases where it is still unclear which of the features survives at finite N, and to what extent. One such problem is the multicritical behavior of $O(N)$ models *at critical dimensions* [7]. Here, one finds that in the $N \to \infty$ limit, there exists a non-trivial UV fixed point, scale invariance is spontaneously broken, and the one parameter family of ground states contains a massive vector and a massless bound state, a Goldstone boson-dilaton. However, since it is unclear whether this structure is likely to survive for finite N [8], one would like to know whether it is possible to construct a local field theory of a massless dilaton via the double scaling limit, where all orders in $1/N$ contribute. The double scaling limit is viewed as the limit at which the attraction between the $O(N)$ vector quanta reaches a value at $g \to g_c$, at which a massless bound state is formed in the $N \to \infty$ limit, while the mass of the vector particle stays finite. In this limit, powers of $1/N$ are compensated by IR singularities and thus all orders in $1/N$ contribute.

In section 2 the double scaling limit for simple integrals and quantum mechanics is recalled, introducing a formalism which will be useful for field

theory examples.

In section 3 the special case of field theory in dimension two is discussed, slightly generalizing previous established results.

In higher dimensions a new phenomenon arises: the possibility of a spontaneous breaking of the $O(N)$ symmetry of the model, associated to the Goldstone phenomenon.

Before discussing a possible double scaling limit, the critical and multi-critical points of the $O(N)$ vector model are reexamined in section 4. In particular, a certain sign ambiguity that appears in the expansion of the gap equation is noted, and related to the existence of the IR fixed point in dimensions $2 < d < 4$. In section 5 we discuss the subtleties and conditions for the existence of an $O(N)$ singlet massless bound state along with a small mass $O(N)$ vector particle excitation. It is pointed out that the correct massless effective field theory is obtained after the massive $O(N)$ scalar is integrated out. Section 6 is devoted to the double scaling limit with a particular emphasis on this limit in theories at their critical dimensions. In section 7 the main conclusions are summarized.

2 Double scaling limit: simple integrals and quantum mechanics

The double scaling limit [1,2,3,4] of the vector model has already been investigated [6], for dimensions $d \leq 1$ and for the simple $(\vec{\phi}^2)^2$ field theory. We first recall results obtained in $d = 0$ and $d = 1$ dimensions, dimensions in which the matrix models have equally been solved. We however introduce a general method, not required here, but useful in the general field theory examples.

2.1 The zero dimensional example

Let us first recall the zero dimensional example. The partition function Z is given by

$$\mathrm{e}^Z = \int \mathrm{d}^N\vec{\phi} \exp\left[-NV\left(\vec{\phi}^2\right)\right].$$

The simplest method for discussing the large N limit is of course to integrate over angular variables (see appendix A1). Instead we introduce two new variables λ, ρ and use the identity

$$\exp\left[-NV(\vec{\phi}^2)\right] \propto \int d\rho\, \mathrm{d}\lambda \exp\left\{-N\left[\tfrac{1}{2}\lambda\left(\vec{\phi}^2-\rho\right)+V(\rho)\right]\right\}. \qquad (2.1)$$

The integral over λ is really a Fourier representation of a δ-function and thus the contour of integration runs parallel to the imaginary axis. The identity

(2.1) transforms the action into a quadratic form in $\vec{\phi}$. Hence the integration over $\vec{\phi}$ can be performed and the dependence in N becomes explicit

$$\mathrm{e}^{Z} \propto \int d\rho\, \mathrm{d}\lambda \exp\left\{-N\left[-\tfrac{1}{2}\lambda\rho + V(\rho) + \tfrac{1}{2}\ln\lambda\right]\right\}.$$

The large N limit is obtained by steepest descent. The saddle point is given by

$$V'(\rho) = \tfrac{1}{2}\lambda\,, \qquad \rho = 1/\lambda\,.$$

The leading contribution to Z is proportional to N and obtained by replacing λ, ρ by the saddle point value. The leading correction is obtained by expanding λ, ρ around the saddle point and performing the gaussian integration. It involves the determinant D of the matrix $\mathbf{M}$ of second derivatives

$$\mathbf{M} = \begin{pmatrix} -\frac{1}{2}\lambda^{-2} & -\frac{1}{2} \\ -\frac{1}{2} & V''(\rho) \end{pmatrix}, \quad D = \det \mathbf{M} = -\tfrac{1}{2}\left(V''(\rho)/\lambda^2 + \tfrac{1}{2}\right).$$

In the generic situation the resulting contribution to Z is $-\frac{1}{2}\ln D$. However if the determinant D vanishes the leading order integral is no longer gaussian, at least for the degree of freedom which corresponds to the eigenvector with vanishing eigenvalue. The condition of vanishing of the determinant also implies that two solutions of the saddle point equation coincide and thus corresponds to a surface in the space of the coefficients of the potential V where the partition function is singular (see appendix A1 for details).

To examine the corrections to the leading large N behaviour it remains however possible to integrate over one of the variables by steepest descent. At leading order this corresponds to solving the saddle point equation for one of the variables, the other being fixed. Here it is convenient to eliminate λ by the equation $\lambda = 1/\rho$. One finds

$$\mathrm{e}^{Z} \propto \int d\rho \exp\left[-N\big(V(\rho) - \tfrac{1}{2}\ln\rho\big) + O(1)\right].$$

In the leading term we obviously recover the result of the angular integration with $\rho = \vec{\phi}^2$. For N large the leading contribution arises from the leading term in the expansion of $W(\rho) = V(\rho) - \frac{1}{2}\ln\rho$ near the saddle point:

$$W(\rho) - W(\rho_s) \sim \tfrac{1}{n!}W^{(n)}(\rho_s)(\rho-\rho_s)^n.$$

The integer n characterizes the nature of the critical point. Adding relevant perturbations $\delta_k V$ of parameters v_k to the critical potential

$$\delta_k V = v_k(\rho - \rho_s)^k, \quad 1 \le k \le n-2$$

(the term $k=n-1$ can always be eliminated by a shift of ρ) we find the partition function at leading order for N large in the scaling region:

$$\mathrm{e}^{Z(\{u_k\})} \propto \int \mathrm{d}z \, \exp\left(-z^n - \sum_{k=1}^{n-2} u_k z^k\right),$$

where $z \propto N^{1/n}(\rho-\rho_s)$ and

$$u_k \propto N^{1-k/n} v_k$$

is held fixed.

2.2 Quantum mechanics

The method we have used above immediately generalizes to quantum mechanics, although a simpler method involves the Schrödinger equation. We consider the euclidean action

$$S(\vec{\phi}) = N \int \mathrm{d}t \, \left[\tfrac{1}{2}\left(\mathrm{d}_t \vec{\phi}(t)\right)^2 + V(\vec{\phi}^2)\right]. \tag{2.2}$$

Note the unusual field normalization, the factor N in front of the action simplifying all expressions in the large N limit.

To explore the large N limit one has to take the scalar function $\vec{\phi}^2$, which self-averages, as a dynamical variable [9]. At each time t we thus perform the transformation (2.1). One introduces two paths $\rho(t), \lambda(t)$ and writes

$$\begin{aligned} &\exp\left[-N \int \mathrm{d}t \, V(\vec{\phi}^2)\right] \\ &\qquad \propto \int [\mathrm{d}\rho(t)\, \mathrm{d}\lambda(t)] \exp\left\{-N \int \mathrm{d}t \left[\tfrac{1}{2}\lambda\left(\vec{\phi}^2 - \rho\right) + V(\rho)\right]\right\}. \end{aligned} \tag{2.3}$$

The integral over the path $\vec{\phi}(t)$ is then gaussian and can be performed. One finds

$$e^Z = \int \mathrm{d}\rho(t)\mathrm{d}\lambda(t) \, \exp\left[-S_{\text{eff}}(\lambda,\rho)\right]$$

with

$$S_{\text{eff}} = N \int \mathrm{d}t \, \left[-\tfrac{1}{2}\lambda\rho + V(\rho) + \tfrac{1}{2}\mathrm{tr}\ln\left(-\mathrm{d}_t^2 + \lambda(t)\right)\right].$$

Again, in the large N limit the path integral can be calculated by steepest descent. The saddle points are constant paths solution of

$$V'(\rho) = \tfrac{1}{2}\lambda\,, \qquad \rho = \frac{1}{2\pi}\int \frac{\mathrm{d}\omega}{\omega^2+\lambda} = \frac{1}{2\sqrt{\lambda}}\,,$$

where ω is the Fourier energy variable conjugate to t. Again a critical point is defined by the property that at least two solutions to the saddle point equations coalesce. This happens when the determinant of the matrix of second derivatives of the equations vanishes:

$$\det\begin{pmatrix} V''(\rho) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4\pi}\int\frac{\mathrm{d}\omega}{(\omega^2+\lambda)^2}\end{pmatrix} = 0\,.$$

The leading correction to the saddle point contribution is given by a gaussian integration. The result involves determinant of the operator second derivative of S_{eff}. By Fourier transforming time the operator becomes a tensor product of 2×2 matrices with determinant $D(\omega)$

$$D(\omega) = \det\begin{pmatrix} V''(\rho) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4\pi}\int\frac{\mathrm{d}\omega'}{(\omega'^2+\lambda)[(\omega-\omega')^2+\lambda]}\end{pmatrix}.$$

Thus, the criticality condition is equivalent to $D(0) = 0$. When the criticality condition is satisfied, the leading correction is no longer given by steepest descent. Again, since at most one mode can be critical, we can integrate over one of the path by steepest descent, which means solving the saddle point equation for one function, the other being fixed. While the ρ-equation remains local, the λ is now non-local, involving the diagonal matrix element of the inverse of the differential operator $-\mathrm{d}_t^2 + \lambda(t)$. We shall see in next section how this problem can be overcome in general. A special feature of quantum mechanics, however, is that the determinant can be calculated, after a simple change of variable. We set

$$\lambda(t) = \dot{s}(t) + s^2(t),$$

in such a way that the second order differential operator factorizes

$$-\mathrm{d}_t^2 + \lambda(t) = -\big(\mathrm{d}_t + s(t)\big)\big(\mathrm{d}_t - s(t)\big). \tag{2.4}$$

The determinant of a first order differential operator can be calculated

$$\ln\det\big(-\mathrm{d}_t^2 + \lambda(t)\big) = \int \mathrm{d}t\, s(t).$$

The jacobian of the transformation (2.4) contributes at higher order in $1/N$ and can be neglected. Therefore the effective action becomes

$$\begin{aligned} S_{\mathrm{eff}} &= N\int \mathrm{d}t\ \left[-\tfrac{1}{2}(\dot{s}+s^2)\rho + V(\rho) + \tfrac{1}{2}s(t)\right] \\ &= N\int \mathrm{d}t\ \left[-\tfrac{1}{2}\rho s^2 + \tfrac{1}{2}s(\dot{\rho}+1) + V(\rho)\right]. \end{aligned}$$

We can now replace s by the solution of a local saddle point equation:

$$-s\rho+\tfrac{1}{2}(\dot\rho+1)=0\,,$$

and find

$$S_{\rm eff}=N\int {\rm d}t\,\left[\frac{\dot\rho^2}{8\rho}+\frac{1}{8\rho}+V(\rho)\right].$$

We recognize the action for the large N potential at zero angular momentum in the radial coordinate $\rho(t)=\vec\phi^2(t)$. Critical points then are characterized by the behaviour of the potential $W(\rho)$

$$W(\rho)=V(\rho)+\frac{1}{8\rho}\,,$$

near the saddle point ρ_s

$$W(\rho)-W(\rho_s)\sim W^{(n)}(\rho_s)\frac{(\rho-\rho_s)^n}{n!}\,.$$

At critical points the ground state energy, after subtraction of the classical term which is linear in N, has a non-analytic contribution. To eliminate N from the action we set

$$t\mapsto tN^{(n-2)/(n+2)},\qquad \rho(t)-\rho_s\mapsto N^{-2/(n+2)}z(t).$$

We conclude that the leading correction to the energy levels is proportional to $N^{-(n-2)/(n+2)}$. Note also that the scaling of time implies that higher order time derivatives would be irrelevant, an observation which can be used more directly to expand the determinant in local terms, and will be important in next section.

If we add relevant corrections to the potential

$$\delta_k V=v_k(\rho-\rho_s)^k,\quad 1\le k\le n-2\,,$$

the coefficients v_k must scale like

$$v_k\propto N^{2(k-n)/(n+2)}.$$

3 The 2D $V(\vec{\phi}^2)$ field theory in the double scaling limit

In the first part we recall the results concerning the $O(N)$ symmetric $V(\vec{\phi}^2)$ field theory, where $\vec{\phi}$ is N-component field, in the large N limit in dimension two because phase transitions occur in higher dimensions, a problem which has to be considered separately. The action is:

$$S(\vec{\phi}) = N \int \mathrm{d}^2 x \left\{ \tfrac{1}{2} \left[\partial_\mu \vec{\phi}(x) \right]^2 + V\left(\vec{\phi}^2 \right) \right\}, \tag{3.1}$$

where an implicit cut-off Λ is always assumed below. Whenever the explicit dependence in the cut-off will be relevant we shall assume a Pauli–Villars's type regularization, i.e. the replacement in action (3.1) of $-\vec{\phi}\partial^2\vec{\phi}$ by

$$-\vec{\phi}\partial^2 D(-\partial^2/\Lambda^2)\vec{\phi}\,, \tag{3.2}$$

where $D(z)$ is a positive non-vanishing polynomial with $D(0) = 1$.

As before one introduces two fields $\rho(x)$ and $\lambda(x)$ and uses the identity (2.3). The effective action is then:

$$S_{\text{eff}} = N \int \mathrm{d}^2 x \left[V(\rho) - \tfrac{1}{2}\lambda\rho \right] + \tfrac{1}{2} N \operatorname{tr} \ln(-\Delta + \lambda). \tag{3.3}$$

Again for N large we evaluate the integral by steepest descent. Since the saddle point value λ is the $\vec{\phi}$-field mass squared, we set in general $\lambda = m^2$. With this notation the two equations for the saddle point $m^2, \rho_s = \langle \vec{\phi}^2 \rangle$ are:

$$V'(\rho_s) = \tfrac{1}{2} m^2\,, \tag{3.4a}$$

$$\rho_s = \frac{1}{(2\pi)^2} \int^\Lambda \frac{\mathrm{d}^2 k}{k^2 + m^2}\,, \tag{3.4b}$$

where we have used a short-cut notation

$$\frac{1}{(2\pi)^2} \int^\Lambda \frac{\mathrm{d}^2 k}{k^2 + m^2} \equiv \frac{1}{(2\pi)^2} \int \frac{\mathrm{d}^2 k}{D(k^2/\Lambda^2)k^2 + m^2} \equiv B_1(m^2).$$

For $m \ll \Lambda$ one finds

$$B_1(m^2) = \frac{1}{2\pi} \ln(\Lambda/m) + \frac{1}{4\pi} \ln(8\pi C) + O(m^2/\Lambda^2)$$

$$\ln(8\pi C) = \int_0^\infty \mathrm{d}s \left(\frac{1}{D(s)} - \theta(1-s) \right).$$

As we have discussed in the case of quantum mechanics a critical point is characterized by the vanishing at zero momentum of the determinant of second derivatives of the action at the saddle point. The mass-matrix has then a zero eigenvalue which, in field theory, corresponds to the appearance of a new massless excitation other than $\vec{\phi}$. In order to obtain the effective action for this scalar massless mode we must integrate over one of the fields [10]. In the field theory case the resulting effective action can no longer be written in local form. To discuss the order of the critical point, however, we only need the action for space independent fields, and thus for example we can eliminate λ using the λ saddle point equation.

The effective ρ potential $W(\rho)$ then reads

$$W(\rho) = V(\rho) - \tfrac{1}{2}\int^{\lambda(\rho)} \mathrm{d}\lambda'\, \lambda' \frac{\partial}{\partial \lambda'} B_1(\lambda'), \tag{3.5}$$

where at leading order for Λ large

$$\lambda(\rho) = 8\pi C\Lambda^2\, \mathrm{e}^{-4\pi\rho}\,.$$

The second term in Eq. (3.5) in fact is the kinetic energy contribution to the ground state free energy as can be viewed in an Hartree–Fock variational calculation that becomes exact in the limit of $N \to \infty$ (see for example Ref. [11]) . The expression for the effective action in Eq. (3.5) is correct for any d and will be used also in section 6.

Here we have:

$$W(\rho) = V(\rho) + C\Lambda^2\, \mathrm{e}^{-4\pi\rho} = V(\rho) + \tfrac{1}{8\pi} m^2\, \mathrm{e}^{-4\pi(\rho-\rho_s)}\,.$$

A multicritical point is defined by the condition

$$W(\rho) - W(\rho_s) = O\left((\rho-\rho_s)^n\right) \tag{3.6}$$

This yields the conditions:

$$V^{(k)}(\rho_s) = \tfrac{1}{2}(-4\pi)^{k-1} m^2 \quad \text{for } 1 \le k \le n-1\,.$$

Note that the coefficients $V^{(k)}(\rho_s)$ are the coupling constants renormalized at leading order for N large. If $V(\rho)$ is a polynomial of degree $n-1$ (the minimal polynomial model) the multicritical condition in Eq. (3.6) and Eq. (3.4b)

determines the critical values of $n-1$ renormalized coupling constants as well as ρ_s in terms of m^2.

When the fields are space-dependent it is simpler to eliminate ρ instead, because the corresponding field equation:

$$V'\big(\rho(x)\big) = \tfrac{1}{2}\lambda(x). \tag{3.7}$$

is local. This equation can be solved by expanding $\rho(x)-\rho_s$ in a power series in $\lambda(x)-m^2$:

$$\rho(x) - \rho_s = \frac{1}{2V''(\rho_s)}\big(\lambda(x) - m^2\big) + O\left((\lambda - m^2)^2\right). \tag{3.8}$$

The resulting action for the field $\lambda(x)$ remains non-local but because, as we shall see, adding powers of λ as well as adding derivatives make terms less relevant, only the few first terms of a local expansion of the effective action will be important.

If in the local expansion of the determinant we keep only the two first terms we obtain an action containing at leading order a kinetic term proportional to $(\partial_\mu\lambda)^2$ and the interaction $(\lambda(x)-m^2)^n$:

$$S_{\text{eff}}(\lambda) \sim N\int \mathrm{d}^2x \left[\frac{1}{96\pi m^4}(\partial_\mu\lambda)^2 + \tfrac{1}{n!}S_n\big(\lambda(x)-m^2\big)^n\right],$$

where the neglected terms are of order $(\lambda-m^2)^{n+1}$, $\lambda\partial^4\lambda$, and $\lambda^2\partial^2\lambda$ and

$$S_n = W^{(n)}(\rho_s)[2V''(\rho_s)]^{-n} = W^{(n)}(\rho_s)(-4\pi m^2)^{-n}.$$

Moreover we note that together with the cut-off Λ, m now also acts as a cut-off in the local expansion.

To eliminate the N dependence in the action we have, as in the example of quantum mechanics, to rescale both the field $\lambda - m^2$ and coordinates:

$$\lambda(x) - m^2 = \sqrt{48\pi}m^2 N^{-1/2}\varphi(x)\,, \quad x \mapsto N^{(n-2)/4}x\,. \tag{3.9}$$

We find

$$S_{\text{eff}}(\varphi) \sim \int \mathrm{d}^2x \left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{n!}g_n\varphi^n\right].$$

In the minimal model, where the polynomial $V(\rho)$ has exactly degree $n-1$, we find $g_n = 6(48\pi)^{(n-2)/2}m^2$.

As anticipated we observe that derivatives and powers of φ are affected by negative powers of N, justifying a local expansion. However we also note that the cut-offs (Λ or the mass m) are now also multiplied by $N^{(n-2)/4}$. Therefore the large N limit also becomes a large cut-off limit.

Double scaling limit. The existence of a double scaling limit relies on the existence of IR singularities due to the massless or small mass bound state which can compensate the $1/N$ factors appearing in the large N perturbation theory.

We now add to the action relevant perturbations:

$$\delta_k V = v_k(\rho(x) - \rho_s)^k, \quad 1 \le k \le n-2.$$

Namely, adding to the λ action a sum of terms proportional to $\int \mathrm{d}^2 x(\lambda - m^2)^k$:

$$\delta_k S_{\text{eff}}(\lambda) = N S_k \int \mathrm{d}^2 x\,(\lambda - m^2)^k,$$

where the coefficients S_k are functions of the coefficients v_k. After the rescaling (3.9) these terms become

$$\delta_k S_{\text{eff}}(\varphi) = \tfrac{1}{k!} g_k N^{(n-k)/2} \int \mathrm{d}^2 x\, \varphi^k(x) \quad 1 \le k \le n-2$$

However, unlike quantum mechanics, it is not sufficient to scale the coefficients g_k with the power $N^{(k-n)/2}$ to obtain a finite scaling limit. Indeed perturbation theory is affected by UV divergences, and we have just noticed that the cut-off diverges with N. In two dimensions the nature of divergences is very simple: it is entirely due to the self-contractions of the interaction terms and only one divergent integral appears:

$$\left\langle \varphi^2(x) \right\rangle = \frac{1}{4\pi^2} \int \frac{\mathrm{d}^2 q}{q^2 + \mu^2}\,,$$

where μ is the small mass of the bound state, required as an IR cut-off to define perturbatively the double scaling limit. We can then extract the N dependence

$$\left\langle \varphi^2(x) \right\rangle = \frac{1}{8\pi}(n-2)\ln N + O(1).$$

Therefore the coefficients S_k have also to cancel these UV divergences, and therefore have a logarithmic dependence in N superposed to the natural power obtained from power counting arguments. In general for any potential $U(\varphi)$

$$U(\varphi) =: U(\varphi) : + \left[\sum_{k=1} \frac{1}{2^k k!} \left\langle \varphi^2 \right\rangle^k \left(\frac{\partial}{\partial \varphi} \right)^{2k} \right] : U(\varphi) :,$$

where $:U(\varphi):$ is the potential from which self-contractions have been subtracted (it has been normal-ordered). For example for $n = 3$

$$\varphi^3(x) =: \varphi^3(x) : +3 \left\langle \varphi^2 \right\rangle \varphi(x),$$

and thus the double scaling limit is obtained with the behaviour

$$N g_1 + \frac{1}{16\pi} \ln N \quad \text{and} \quad N^{1/2} g_2 \text{ fixed } .$$

For another example $n = 4$

$$g_1 N^{3/2} \quad \text{and} \quad N g_2 + \frac{g_4}{8\pi} \ln N \text{ fixed } .$$

4 The $V(\vec{\phi}^2)$ in the large N limit: phase transitions

In higher dimensions something new happens: the possibility of phase transitions associated with spontaneous breaking the $O(N)$ symmetry. In the first part we thus study the $O(N)$ symmetric $V(\vec{\phi}^2)$ field theory, in the large N limit to explore the possible phase transitions and identify the corresponding multicritical points. The action is:

$$S(\vec{\phi}) = N \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\partial_\mu \vec{\phi}(x) \right]^2 + V\left(\vec{\phi}^2 \right) \right\}, \tag{4.1}$$

where, as above (Eqs.(3.1)-(3.2)), an implicit cut-off Λ is always assumed below.

The identity (2.3) transforms the action into a quadratic form in $\vec{\phi}$ and therefore the integration over $\vec{\phi}$ can be performed. It is convenient however here to integrate only over $N - 1$ components, to keep a component of the vector field, which we denote σ, in the action. The effective action is then:

$$S_{\text{eff}} = N \int \mathrm{d}^d x \left[\tfrac{1}{2} \left(\partial_\mu \sigma \right)^2 + V(\rho) + \tfrac{1}{2} \lambda \left(\sigma^2 - \rho \right) \right] + \tfrac{1}{2}(N-1) \operatorname{tr} \ln(-\Delta + \lambda). \tag{4.2}$$

4.1 The saddle point equations: the $O(N)$ critical point

Let us then write the saddle point equations for a general potential V. At high temperature $\sigma = 0$ and λ is the $\vec{\phi}$-field mass squared. We thus set in general $\lambda = m^2$. With this notation the three saddle point equations are:

$$m^2 \sigma = 0 \,, \tag{4.3a}$$

$$V'(\rho) = \tfrac{1}{2} m^2 \,, \tag{4.3b}$$

$$\sigma^2 = \rho - \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d k}{k^2 + m^2} \,, \tag{4.3c}$$

with the notation of section 3

$$\frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d k}{k^2 + m^2} \equiv \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{D(k^2/\Lambda^2) k^2 + m^2} \equiv B_1(m^2).$$

In the ordered phase $\sigma \neq 0$ and thus m vanishes. Equation (4.3c) has a solution only for $\rho > \rho_c$,

$$\rho_c = \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d k}{k^2} \,, \ \Rightarrow \ \sigma = \sqrt{\rho - \rho_c} \,.$$

Equation (4.3b) which reduces to $V'(\rho) = 0$ then yields the critical temperature. Setting $V(\rho) = U(\rho) + \frac{1}{2} r \rho$, we find

$$r_c = -2U'(\rho_c).$$

To find the magnetization critical exponent β we need the relation between the r and ρ near the critical point.

In the disordered phase, $\sigma = 0$, equation (4.3c) relates ρ to the $\vec{\phi}$-field mass m. It can be rewritten

$$\rho = B_1(m^2) = \Lambda^{d-2} F(m^2/\Lambda^2), \tag{4.4}$$

where

$$F(z) = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} \int \frac{k^{d-1} dk}{k^2 D(k^2) + z} \,. \tag{4.5}$$

The function $F(z)$ can be written in an asymptotic expansion

$$F(z) = z^{d/2-1} \sum_{n=0}^{\infty} b_n z^n + \sum_{n=0}^{\infty} c_n z^n. \tag{4.6}$$

The non-analytic part can be extracted from the representation

$$F(z) = \frac{z^{d/2-1}}{2\Gamma(d/2)(4\pi)^{d/2}} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, y^{-d/2} x^{d/2-1} \exp\left\{-\frac{x}{2}D(zx/y) - \frac{y}{2}\right\}, \tag{4.7}$$

which gives e.g. in the case of $D(k^2) = 1 + d_1 k^2$ the asymptotic expansion for the non-analytic part:

$$F^{\mathrm{NA}}(z) = \Gamma(1-d/2) \frac{z^{d/2-1}}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \frac{(d_1 z)^n}{n!} \frac{\Gamma(\frac{d}{2}+2n)}{\Gamma(\frac{d}{2}+n)}\,, \tag{4.8}$$

and the analytic part is obtained from $F(z) - F^{\mathrm{NA}}(z)$. For $m \ll \Lambda$, ρ approaches ρ_c,

$$\rho_c = B_1(0) = \Lambda^{d-2} F(0),$$

and the relation becomes:

$$\rho - \rho_c = -K(d) m^{d-2} + a(d) m^2 \Lambda^{d-4} + O\left(m^d \Lambda^{-2}\right) + O\left(m^4 \Lambda^{d-6}\right). \tag{4.9}$$

For $2 < d < 4$ (the situation we shall assume below except when stated otherwise) the $O\left(m^d \Lambda^{-2}\right)$ from the non-analytic part dominates the corrections to the leading part of this expression. For $d = 4$ instead

$$\rho - \rho_c = \tfrac{1}{8\pi^2} m^2 \left(\ln m/\Lambda + \text{ const.}\right),$$

and for $d > 4$ the analytic contribution dominates and

$$\rho - \rho_c \sim a(d) m^2 \Lambda^{d-4}.$$

The constant $K(d)$ is universal:

$$K(d) = -b_0 = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{k^2(k^2+1)} = -\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\,. \tag{4.10}$$

The constant $a(d)$ instead depends on the cut-off procedure, and is given by

$$a(d) = c_1 = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{k^4} \left(1 - \frac{1}{D^2(k^2)}\right). \tag{4.11}$$

Let us also define for later purpose the function

$$B_2(p; m^2) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{\left[k^2 D(k^2/\Lambda^2) + m^2\right]\left[(p-k)^2 D((p-k)^2/\Lambda^2) + m^2\right]}, \tag{4.12}$$

which is up to sign the second derivative with respect to $\lambda(x)$ of the tr ln term in the effective action. Then for m small

$$\begin{aligned} B_2(0;m^2) &= -\frac{\mathrm{d}}{\mathrm{d}m^2} B_1(m^2) = -\frac{\mathrm{d}}{\mathrm{d}m^2}(\rho - \rho_c) \\ &= \tfrac{1}{2}(d-2)K(d)m^{d-4} - a(d)\Lambda^{d-4} + O\left(m^{d-2}\Lambda^{-2}, m^2\Lambda^{d-6}\right). \end{aligned} \tag{4.13}$$

Critical point. In a generic situation $V''(\rho_c) = U''(\rho_c)$ does not vanish. We thus find in the low temperature phase

$$t = r - r_c \sim -2U''(\rho_c)(\rho - \rho_c) \ \Rightarrow\ \beta = \tfrac{1}{2}. \tag{4.14}$$

This is the case of an ordinary critical point. Stability implies $V''(\rho_c) > 0$ so that $t < 0$.

At high temperature, in the disordered phase, the $\vec{\phi}$-field mass m is given by $2U'(\rho) + r = m^2$ and thus, using (4.9), at leading order

$$t \sim 2U''(\rho_c)K(d)m^{d-2},$$

in agreement with the result of the normal critical point. Of course the simplest realization of this situation is to take $V(\rho)$ quadratic, and we recover the $(\vec{\phi}^2)^2$ field theory.

The sign of the constant $a(d)$. A comment concerning the non-universal constant $a(d)$ defined in (4.9) is here in order because, while its absolute value is irrelevant, its sign plays a role in the discussion of multicritical points. Actually this sign is already relevant to the RG properties of the large N limit of simple scalar field theories. In a $V(\vec{\phi}^2) = \frac{\mu}{2}\vec{\phi}^2 + \frac{\lambda}{4!}(\vec{\phi}^2)^2$ theory (in Eq. (4.1)) it is easy to verify that $a(d)$ is related to the second coefficient of the large N RG β-function. If we call $N\lambda = g\Lambda^{4-d}$ the bare coupling constant, we indeed find [9]:

$$\beta(g) = -(4-d)g + \tfrac{1}{6}(4-d)Na(d)g^2 + O\left(1/N\right). \tag{4.15}$$

It is generally assumed that $a(d) > 0$. Indeed, this is what is found near four dimensions in all regularizations. Then there exists an IR fixed point, non-trivial zero of the β-function. For the simplest Pauli–Villars's type regularization we have $D(z) > 1$ and thus $a(d)$ is finite and positive in dimensions $2 < d < 4$, but this is not a universal feature. Even in case of simple lattice regularizations it has been shown [12] that in $d = 3$ the sign is arbitrary. We

illustrate the ambiguity in the sign of $a(d)$ in $2 < d < 4$ by explicit examples in appendix A2. However, if $a(d)$ is negative the large N RG has a problem, since the coupling flows in the IR limit to large values where the large N expansion is no longer reliable. It is not known whether this signals a real physical problem, or is just an artifact of the large N limit.

Another way of stating the problem is to examine directly the relation between bare and renormalized coupling constant. Calling $g_{\mathrm{r}} m^{4-d}$ the renormalized 4-point function at zero momentum, we find

$$m^{4-d} g_{\mathrm{r}} = \frac{\Lambda^{4-d} g}{1 + \Lambda^{4-d} g N B_2(0; m^2)/6}. \tag{4.16}$$

In the limit $m \ll \Lambda$ the relation can be written

$$\frac{1}{g_{\mathrm{r}}} = \frac{(d-2)NK(d)}{12} + \left(\frac{m}{\Lambda}\right)^{4-d} \left(\frac{1}{g} - \frac{Na(d)}{6}\right). \tag{4.17}$$

We see that when $a(d) < 0$ the renormalized IR fixed point value cannot be reached by varying $g > 0$ for any finite value of m/Λ.

4.2 Multicritical points

A new situation arises if we can adjust a parameter of the potential in such a way that $U''(\rho_c) = 0$. This can be achieved only if the potential V is at least cubic. We then expect a tricritical behavior. Higher critical points can be obtained when more derivatives vanish. We shall examine the general case though, from the point of view of real phase transitions, higher order critical points are not interesting because $d > 2$ for continuous symmetries and mean-field behavior is then obtained for $d \geq 3$. The analysis will however be useful in the study of double scaling limit.

Assuming that the first non-vanishing derivative is $U^{(n)}(\rho_c)$, we expand further equation (4.3b). In the ordered low temperature phase we now find

$$t = -\frac{2}{(n-1)!} U^{(n)}(\rho_c)(\rho - \rho_c)^{n-1}, \ \Rightarrow \ \sigma \propto (-t)^{\beta}, \quad \beta = \frac{1}{2(n-1)}, \tag{4.18}$$

which leads to the exponent β expected in the mean field approximation for such a multicritical point. We have in addition the condition $U^{(n)}(\rho_c) > 0$.

In the high temperature phase instead

$$m^2 = t + (-1)^{n-1} \frac{2}{(n-1)!} U^{(n)}(\rho_c) K^{n-1}(d) m^{(n-1)(d-2)}. \tag{4.19}$$

For $d > 2n/(n-1)$ we find a simple mean field behavior, as expected since we are above the upper-critical dimension .

For $d < 2n/(n-1)$ we find a peculiar phenomenon, the term in the r.h.s. is always dominant, but depending on the parity of n the equation has solutions for $t > 0$ or $t < 0$. For n even, t is positive and we find

$$m \propto t^{\nu}, \qquad \nu = \frac{1}{(n-1)(d-2)}, \tag{4.20}$$

which is a non mean-field behavior below the critical dimension. However for n odd (this includes the tricritical point) t must be negative, in such a way that we have now two competing solutions at low temperature. We have to find out which one is stable. We shall verify below that only the ordered phase is stable, so that the correlation length of the $\vec{\phi}$-field in the high temperature phase always remains finite. Although these dimensions do not correspond to physical situations because $d < 3$ the result is peculiar and inconsistent with the ε-expansion.

For $d = 2n/(n-1)$ we find a mean field behavior without logarithmic corrections, provided one condition is met:

$$\frac{2}{(n-1)!} U^{(n)}(\rho_c) K^{n-1} \left(2n/(n-1)\right) < 1\,, \qquad K(3) = 1/(4\pi). \tag{4.21}$$

We examine, as an example, in more details the tricritical point below. We will see that the special point

$$\frac{2}{(n-1)!} U^{(n)}(\rho_c) K^{n-1} \left(2n/(n-1)\right) = 1\,, \tag{4.22}$$

has several peculiarities [7]. In what follows we call Ω_c this special value of $U^{(n)}(\rho_c)$.

Discussion. In the mean field approximation the function $U(\rho) \propto \rho^n$ is not bounded from below for n odd, however $\rho = 0$ is the minimum because by definition $\rho \geq 0$. Here instead we are in the situation where $U(\rho) \sim (\rho - \rho_c)^n$ but ρ_c is positive. Thus this extremum of the potential is likely to be unstable for n odd. To check the global stability requires further work. The question is whether such multicritical points can be studied by the large N limit method.

Another point to notice concerns renormalization group: The $n = 2$ example is peculiar in the sense that the large N limit exhibits a non-trivial IR fixed point. For higher values of n no coupling renormalization arises in

the large N limit and the IR fixed point remains pseudo-gaussian. We are in a situation quite similar to usual perturbation theory, the β function can only be calculated perturbatively in $1/N$ and the IR fixed point is outside the perturbative regime.

4.3 Local stability and the mass matrix

The matrix of the general second partial derivatives of the effective action is:

$$N \begin{pmatrix} p^2 + m^2 & 0 & \sigma \\ 0 & V''(\rho) & -\frac{1}{2} \\ \sigma & -\frac{1}{2} & -\frac{1}{2} B_2(p; m^2) \end{pmatrix}, \tag{4.23}$$

where $B_2(p; m^2)$ is defined in (4.12).

We are in position to study the local stability of the critical points. Since the integration contour for $\lambda = m^2$ should be parallel to the imaginary axis, a necessary condition for stability is that the determinant remains negative.

The disordered phase. Then $\sigma = 0$ and thus we have only to study the 2×2 matrix $\mathbf{M}$ of the ρ, m^2 subspace. Its determinant must remain negative which implies

$$\det \mathbf{M} < 0 \;\Leftrightarrow\; 2V''(\rho) B_2(p; m^2) + 1 > 0\,. \tag{4.24}$$

For Pauli–Villars's type regularization the function $B_2(p; m^2)$ is decreasing so that this condition is implied by the condition at zero momentum

$$\det \mathbf{M} < 0 \;\Leftarrow\; 2V''(\rho) B_2(0; m^2) + 1 > 0\,.$$

For m small we use Eq. (4.13) and at leading order the condition becomes:

$$K(d)(d-2) m^{d-4} V''(\rho) + 1 > 0\,.$$

This condition is satisfied by a normal critical point since $V''(\rho_c) > 0$. For a multicritical point, and taking into account equation (4.9) we find:

$$(-1)^n \frac{d-2}{(n-2)!} K^{n-1}(d) m^{n(d-2)-d} V^{(n)}(\rho_c) + 1 > 0\,. \tag{4.25}$$

We obtain a result consistent with our previous analysis: For n even it is always satisfied. For n odd it is always satisfied above the critical dimension and never below. At the upper-critical dimension we find a condition on

the value of $V^{(n)}(\rho_c)$ which we recognize to be identical to condition (4.21) because then $2/(n-1) = d-2$.

The ordered phase. Now $m^2 = 0$ and the determinant Δ of the complete matrix is:

$$-\Delta > 0 \;\Leftrightarrow\; 2V''(\rho)B_2(0;p^2)p^2 + p^2 + 4V''(\rho)\sigma^2 > 0\,. \tag{4.26}$$

We recognize a sum of positive quantities, and the condition is always satisfied. Therefore in the case where there is a competition with a disordered saddle point only the ordered one can be stable.

5 The scalar bound state

In this section we study the limit of stability in the disordered phase ($\sigma = 0$). This is a problem which only arises when n is odd, the first case being provided by the tricritical point. The mass-matrix has then a zero eigenvalue which corresponds to the appearance of a new massless excitation other than $\vec{\phi}$. Let us denote by $\mathbf{M}$ the ρ, m^2 2×2 submatrix. Then

$$\det \mathbf{M} = 0 \;\Leftrightarrow\; 2V''(\rho)B_2(0;m^2) + 1 = 0\,.$$

In the two-space the corresponding eigenvector has components $(\frac{1}{2}, V''(\rho))$.

5.1 The small mass m region

In the small m limit the equation can be rewritten in terms of the constant $K(d)$ defined in (4.10):

$$K(d)(d-2)m^{d-4}V''(\rho) + 1 = 0\,. \tag{5.1}$$

Equation (5.1) tells us that $V''(\rho)$ must be small. We are thus close to a multicritical point. Using the result of the stability analysis we obtain

$$(-1)^{n-1}\frac{d-2}{(n-2)!}K^{n-1}(d)m^{n(d-2)-d}V^{(n)}(\rho_c) = 1\,. \tag{5.2}$$

We immediately notice that this equation has solutions only for $n(d-2) = d$, i.e. at the critical dimension. The compatibility then fixes the value of $V^{(n)}(\rho_c)$. We again find the point (4.22), $V^{(n)}(\rho_c) = \Omega_c$. If we take into account the leading correction to the small m behavior we find instead:

$$V^{(n)}(\rho_c)\Omega_c^{-1} - 1 \sim (2n-3)\frac{a(d)}{K(d)}\left(\frac{m}{\Lambda}\right)^{4-d}. \tag{5.3}$$

This means that when $a(d) > 0$ there exists a small region $V^{(n)}(\rho_c) > \Omega_c$ where the vector field is massive with a small mass m and the bound-state massless. The value Ω_c is a fixed point value.

The scalar field at small mass. We want to extend the analysis to a situation where the scalar field has a small but non-vanishing mass M and m is still small. The goal is in particular to explore the neighborhood of the special point (4.22). Then the vanishing of the determinant of $\mathbf{M}$ implies

$$1 + 2V''(\rho)B_2(iM; m^2) = 0 \,. \tag{5.4}$$

Because M and m are small, this equation still implies that ρ is close to a point ρ_c where $V''(\rho)$ vanishes. Since reality imposes $M < 2m$, it is easy to verify that this equation has also solutions for only the critical dimension. Then

$$V^{(n)}(\rho_c) f(m/M) = \Omega_c \,, \tag{5.5}$$

where we have set:

$$f(z) = \int_0^1 \mathrm{d}x \left[1 + (x^2 - 1)/(4z^2)\right]^{d/2-2}, \qquad \tfrac{1}{2} < z \,. \tag{5.6}$$

In three dimensions it reduces to:

$$f(z) = z \ln\left(\frac{2z+1}{2z-1}\right) .$$

$f(z)$ is a decreasing function which diverges for $z = \frac{1}{2}$ because $d \le 3$. Thus we find solutions in the whole region $0 < V^{(n)}(\rho_c) < \Omega_c$, i.e. when the multicritical point is locally stable.

Let us calculate the propagator near the pole. We find the matrix $\mathbf{\Delta}$

$$\mathbf{\Delta} = \frac{2}{G^2}\left[N \left.\frac{\mathrm{d}B_2(p; m^2)}{\mathrm{d}p^2}\right|_{p^2 = -M^2}\right]^{-1} \frac{1}{p^2 + M^2} \begin{pmatrix} 1 & G \\ G & G^2 \end{pmatrix}, \tag{5.7}$$

where we have set

$$G = \frac{2(-K)^{n-2} W^{(n)}}{(n-2)!} m^{4-d} \,.$$

For m/M fixed the residue goes to zero with m as m^{d-2} because the derivative of B is of the order of m^{d-6}. Thus the bound-state decouples on the multicritical line.

5.2 The scalar massless excitation: general situation

Up to now we have explored only the case where both the scalar field and the vector field propagate. Let us now relax the latter condition, and examine what happens when m is no longer small. The condition $M = 0$ then reads

$$2V''(\rho_s)B_2(0; m^2) + 1 = 0$$

together with

$$m^2 = 2V'(\rho_s), \qquad \rho_s = B_1(m^2)\,. \tag{5.8}$$

An obvious remark is: there exist solutions only for $V''(\rho_s) < 0$, and therefore the ordinary critical line can never be approached. In terms of the function $F(z)$ defined by equation (4.4) the equations can be rewritten

$$\rho_s = \Lambda^{d-2}F(z), \quad z = 2V'(\rho_s)\Lambda^{-2}, \quad 2\Lambda^{d-4}V''(\rho_s)F'(z) = 1\,.$$

The function $F(z)$ in Pauli–Villars's regularization is a decreasing function. In the same way $-F'(z)$ is a positive decreasing function.

The third equation is the condition for the two curves corresponding to the two first ones become tangent. For any value of z we can find potentials and thus solutions. Let us call z_s such a value and specialize to cubic potentials. Then

$$\rho_s = \Lambda^{d-2}F(z_s) \quad ,$$

$$V(\rho) = V'(\rho_s)(\rho-\rho_s) + \tfrac{1}{2}V''(\rho_s)(\rho-\rho_s)^2 + \tfrac{1}{3!}V^{(3)}(\rho_s)(\rho-\rho_s)^3, \tag{5.9}$$

which yields a two parameter family of solutions. For z small we see that for $d < 4$ the potential becomes proportional to $(\rho-\rho_c)^3$.

6 Stability and double scaling limit

In order to discuss in more details the stability issue and the double scaling limit we now construct the effective action for the scalar bound state. We consider first only the massless case. We only need the action in the IR limit, and in this limit we can integrate out the vector field and the second massive eigenmode.

Integration over the massive modes. As we have already explained in section 3 we can integrate over one of the fields, the second being fixed, and we need only the result at leading order. Therefore we replace in the functional integral

$$\mathrm{e}^Z = \int [\mathrm{d}\rho\mathrm{d}\lambda] \exp\left[-\tfrac{N}{2}\mathrm{tr}\ln(-\partial^2+\lambda) + N\int \mathrm{d}^d x\left(-V(\rho)+\tfrac{1}{2}\rho\lambda\right)\right], \tag{6.1}$$

one of the fields by the solution of the field equation. It is useful to discuss the effective potential of the massless mode first. This requires calculating the action only for constant fields it is then simpler to eliminate λ. We assume in this section that m is small (the vector propagates). For $\lambda \ll \Lambda$ the λ-equation reads ($d < 4$)

$$\rho - \rho_c = -K(d)\lambda^{(d-2)/2}. \tag{6.2}$$

It follows that the resulting potential $W(\rho)$, obtained from Eq. (3.5) is

$$W(\rho) = V(\rho) + \frac{d-2}{2d(K(d))^{2/(d-2)}}(\rho_c - \rho)^{d/(d-2)}. \tag{6.3}$$

In the sense of the double scaling limit the criticality condition is

$$W(\rho) = O\big((\rho - \rho_s)^n\big).$$

It follows

$$V^{(k)}(\rho_s) = -\tfrac{1}{2}K^{1-k}(d)\frac{\Gamma\big(k - d/(d-2)\big)}{\Gamma\big(-2/(d-2)\big)}m^{d-k(d-2)} \quad 1 \le k \le n-1\,.$$

For the potential V of minimal degree we find

$$W(\rho) \sim \tfrac{1}{2n!}K^{1-n}(d)\frac{\Gamma\big(n - d/(d-2)\big)}{\Gamma\big(-2/(d-2)\big)}m^{d-n(d-2)}(\rho - \rho_s)^n.$$

The double scaling limit. We recall here that quite generally one verifies that a non-trivial double scaling limit may exist only if the resulting field theory of the massless mode is super-renormalizable, i.e. below its upper-critical dimension $d = 2n/(n-2)$, because perturbation theory has to be IR divergent. Equivalently, to eliminate N from the critical theory, one has to rescale

$$\rho - \rho_s \propto N^{-2\theta}\varphi\,, \quad x \mapsto xN^{(n-2)\theta} \quad \text{with } 1/\theta = 2n - d(n-2),$$

where θ has to be positive.

We now specialize to dimension three, since $d < 3$ has already been examined, and the expressions above are valid only for $d < 4$. The normal critical point ($n = 3$), which leads to a φ^3 field theory, and can be obtained for a quadratic potential $V(\rho)$ (the $(\vec{\phi}^2)^2$) has been discussed elsewhere [6]. We

thus concentrate on the next critical point $n = 4$ where the minimal potential has degree three.

The $d = 3$ tricritical point. The potential $W(\rho)$ then becomes

$$W(\rho) = V(\rho) + \tfrac{8\pi^2}{3}(\rho_c - \rho)^3. \tag{6.4}$$

If the potential $V(\rho)$ has degree larger than three, we obtain after a local expansion and a rescaling of fields,

$$\rho - \rho_s = (\frac{-1}{32\pi^2 \rho_c})(\lambda - m^2) \propto \varphi/N\,, \quad x \mapsto Nx\,, \tag{6.5}$$

a simple super-renormalizable $\varphi^4(x)$ field theory. If we insist instead that the initial theory should be renormalizable, then we remain with only one candidate, the renormalizable $(\vec{\phi}^2)^3$ field theory, also relevant for the tricritical phase transition with $O(N)$ symmetry breaking. Inspection of the potential $W(\rho)$ immediately shows a remarkable feature: Because the term added to $V(\rho)$ is itself a polynomial of degree three, the critical conditions lead to a potential $W(\varphi)$ which vanishes identically. This result reflects the property that the two saddle point equations ($\partial S/\partial\rho = 0$, $\partial S/\partial\lambda = 0$ in Eqs. (4.3)) are proportional and thus have a continuous one-parameter family of solutions. This results in a flat effective potential for $\varphi(x)$. The effective action for φ depends only on the derivatives of φ, like in the $O(2)$ non-linear σ model. We conclude that no non-trivial double scaling limit can be obtained* under these conditions. In three dimensions with a $(\vec{\phi}^2)^3$ interaction we can generate at most a normal critical point $n = 3$, but then a simple $(\vec{\phi}^2)^2$ field theory suffices.

The ambiguity of the sign of $a(d)$ discussed in section 4 and in Appendix A2 has an interesting appearance in $d = 3$ in the small m^2 region. If one keeps the extra term proportional to $a(d)$ in Eq. (6.3) we have

$$W(\rho) = V(\rho) + \frac{8\pi^2}{3}(\rho_c - \rho)^3 + \frac{a(3)}{\Lambda}4\pi^2(\rho_c - \rho)^4.$$

Using now Eq. (6.2) and, as mentioned in section 5, the fact that in the small m^2 region the potential is proportional to $(\rho - \rho_c)^3$ we can solve for m^2. Since

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* It was pointed out that another sort of obstacle is present for $(\vec{\phi}^2)^2$ model in $d = 2$ [13] and $d = 4$ [14].

$m^2 > 0$ the appearance of a phase with small mass depends on the sign of $a(d)$. Clearly this shows a non-commutativity of the limits of $m^2/\Lambda^2 \to 0$ and $N \to \infty$. The small m^2 phase can be reached by a special tuning and cannot be reached with an improper sign of $a(d)$. Calculated in this way, m^2 can be made proportional to the deviation of the coefficient of ρ^3 in $V(\rho)$ from its critical value $16\pi^2$.

7 Conclusions

This is a study of several subtleties in the phase structure of $O(N)$ vector models around multicritical points of odd and even orders. One of the main topics is the understanding of the multicritical behavior of these models at their critical dimensions and the effective field theory of the $O(N)$-singlet bound state obtained in the $N \to \infty,\;\; g \to g_c$ correlated limit. It is pointed out (in contrast to previous studies) that the integration over massive $O(N)$ singlet modes is essential in order to extract the correct effective field theory of the small mass scalar excitation. After performing this integration, it has been established here that the double scaling limit of $(\vec{\phi}^2)^n$ vector model in its critical dimension $d = 2n/(n-1)$ can result in a theory of a free massless $O(N)$ singlet bound state. This fact is a consequence of the existence of flat directions at the scale invariant multicritical point in the effective action. In contrast to the case $d < 2n/(n-1)$ where IR singularities compensate powers of $1/N$ in the double scaling limit, at $d = 2n/(n-1)$ there is no such compensation and only a noninteracting effective field theory of the massless bound state is left.

Another interesting issue in this study is the ambiguity of the sign of $a(d)$. The coefficient of $m^2\Lambda^{d-4}$ denoted by $a(d)$ in the expansion of the gap equation in Eqs. (4.3c) and (4.9) seems to have a surprisingly important role in the approach to the continuum limit ($\Lambda^2 \gg m^2$). The existence of an IR fixed point at $g \sim O(N^{-1})$, as seen in the β function for the unrenormalized coupling constant in Eq. (4.15), depends on the sign of $a(d)$. Moreover, as seen in section 3.1 the existence of a phase with a small mass m for the $O(N)$ vector quanta and a massless $O(N)$ scalar depends also on the sign of $a(d)$. It may very well be that the importance of the sign of $a(d)$ is a mere reflection of the limited coupling constant space used to described the model. This is left here as an open question that deserves a detailed renormalization group or lattice simulation study in the future.

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Appendices

A1 Double scaling limit: $d = 0$

We consider the sum Z of connected Feynman diagrams in $d = 0$ dimension:

$$\mathrm{e}^{Z} = \int \mathrm{d}^{N}\vec{\phi}\,\mathrm{e}^{-NV(\vec{\phi}^{2})} = \frac{\pi^{N/2}}{\Gamma(N/2)}\int_{0}^{\infty}\frac{\mathrm{d}x}{x}\,\mathrm{e}^{-N[V(x)-\ln(x)/2]}, \qquad (A1.1)$$

where x is normalized by $V(x) = \frac{1}{2}x + O(x^2)$.

We define

$$W(x) = V(x) - \tfrac{1}{2}\ln(x).$$

The saddle point equation reads

$$W'(x_s) = 0\,.$$

A critical point of order n is defined by the conditions:

$$W(x) - W(x_s) \underset{x\to x_s}{\sim} W^{(n)}(x_s)\frac{(x-x_s)^n}{n!}$$

The critical potential $V_n(x)$ of lowest degree thus satisfies

$$V_n'(x) = \left[1-(1-x/(n-1))^{n-1}\right]/(2x),$$

and therefore

$$V_n(x) = \tfrac{1}{2}\int_0^{x/(n-1)}\frac{\mathrm{d}y}{y}\left[1-(1-y)^{n-1}\right].$$

and

$$W_n' \underset{x\to n}{\sim} -\tfrac{1}{2(n-1)}(1-x/(n-1))^{n-1}.$$

If we now change the variable, setting

$$(1 - x/(n-1))N^{1/n} = z\,, \tag{A1.2}$$

we find

$$\mathrm{e}^{Z_c} \sim \frac{\pi^{N/2}N^{-1/n}}{\Gamma(N/2)} \exp[\tfrac{1}{2}N(\ln(n-1)-1-\tfrac{1}{2}-\cdots-\tfrac{1}{n-1})] \int \mathrm{d}z\ \mathrm{e}^{-z^n/2n+O(N^{-1/n}z^{n+1})}\,.$$

Adding to the critical potential all relevant perturbations, and shifting the saddle point back to $x = n-1$, we obtain

$$V(x) = V_n(x) + \sum_{k=1}^{n-1} \frac{v_k}{2k}\left[(1 - x/(n-1))^k - \tfrac{k}{n}(1 - x/(n-1))^n\right]. \tag{A1.3}$$

After the change of variable ($A1.2$), we find at leading order for N large

$$N\big(W(x) - W(n)\big) = \tfrac{1}{2n}z^n + \sum_{k=1}^{n-1} \frac{v_k}{2k}\left[N^{1-k/n}z^k - \tfrac{k}{n}z^n\right].$$

We see that a double scaling limit is reached only when the coefficients v_k go to zero for large N with $v_k N^{1-k/n} = u_k$ fixed. This implies that the coupling constants, i.e. the coefficients of $V(x)$ in the expansion in powers of x must approach the critical values corresponding to the potential V_n with a well-defined behaviour as a function of N. Note also that since for all $v_k = 0$ several solutions of the saddle point equation coalesce, the critical potential corresponds to a singularity of the large N partition function in the space of coupling constants. Actually if we expand Z in powers of $1/N$,

$$Z = \sum_{h=1} N^{1-h} Z_h\,,$$

all terms Z_h are singular at this point. This can be most easily seen by keeping only the most relevant term proportional to v_1. Then one finds

$$Z_h \propto v_1^{(1-h)n/(n-1)}.$$

The scaling behaviour of v_1 as a function of N uses this singular behaviour to compensate the powers of N. Finally by tuning the coupling constants to reach a singularity we approach the radius of convergence of the perturbative expansion at h fixed, and thus enhance high order Feynman diagrams. Thus

one reaches a dense configuration of contributing Feynman graphs which is the analog of the sum over surfaces in $O(N)$ matrix models. In the $O(N)$ vector model this is an expansion over 'randomly branched polymers'. The scaling partition function is obtained by simplifying the integrand

$$N\bigl(W(x)-W(n-1)\bigr) \sim \tfrac{1}{2n} z^n + \sum_{k=1}^{n-2} \frac{u_k}{2k} z^k ,$$

because the v_k goes to zero and the term $k=n-1$ can be eliminated by shifting z. Finally, if we keep only the most relevant perturbation term proportional to v_1 the double scaled partition function is given by a generalized Airy function

$$Z(u_1) = \int \mathrm{d}z \exp\left(-\tfrac{1}{2}\left(\tfrac{z^n}{n} + u_1 z\right)\right).$$

A2 Regularization and sign of the β-function

Here we give a few explicit examples which show the regularization dependence of the non-universal constant $a(d)$ defined in (4.9).

Pauli–Villars regularization. For the regularization (3.2) it is given by

$$a(d) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{k^4} \left(1 - \frac{1}{D^2(k^2)}\right).$$

Let us consider for example

$$D(k^2) = 1 + \alpha k^2 + \beta k^4 .$$

$D(k^2) > 0$ implies $4\beta > \alpha^2$.

If $\alpha > 0$ then clearly for every k, $D(k^2) > 1$, and $a(d)$ is positive. However if $\alpha < 0$ it is possible to choose the parameters in such a way that $a(d)$ will change sign; especially if one takes α close to $-2\sqrt{\beta}$. Possible values of $a(d)$ are exhibited in Fig. 1 below.

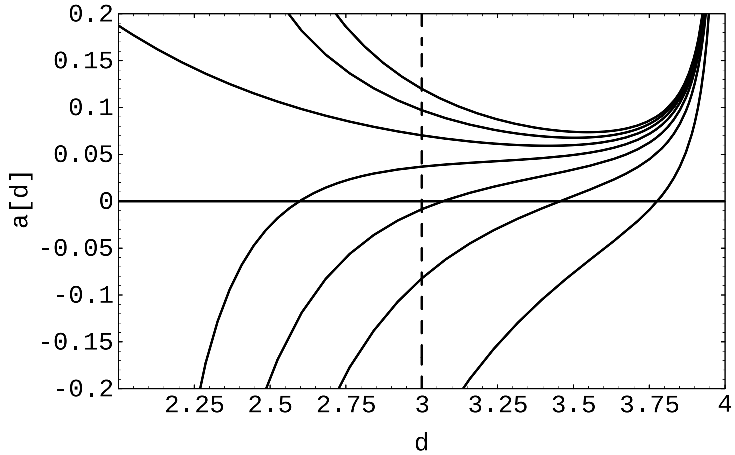

Fig. 1: Values of $a(d)$ in Pauli–Villars regularization.
(From top to bottom: $\alpha = 0.8,\ 0.4,\ 0,\ -0.4,\ -0.8,\ -1.2,\ -1.6$ and $\beta = 1$.)

Lattice regularization. We extend here the calculations of [12] to more general lattice regularizations, to understand how general this sign property is.

An alternative representation to Eq. (4.4) is:

$$\rho = \tfrac{1}{2}\int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{-\alpha m^2/2}\, f(\alpha)$$
$$f(\alpha) = \int_{-\pi}^{\pi} \tfrac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{-\alpha Q(k)/2},$$

where $(Q(k))^{-1}$ is the massless free field propagator. Using the asymptotics of $f(\alpha)$

$$f(\alpha) \underset{\alpha\to\infty}{\sim} (2\pi\alpha)^{-\frac{d}{2}} \left[1 + O(\tfrac{1}{\alpha})\right],$$

then

$$\rho - \rho_c = -K(d) m^{d-2} + \tfrac{1}{2}\int_0^\infty \mathrm{d}\alpha \left(\mathrm{e}^{-\alpha m^2/2} - 1\right)\left(f(\alpha) - (2\pi\alpha)^{-d/2}\right).$$

Therefore $a(d)$ in Eq.(4.9) reads:

$$a(d) = -\tfrac{1}{4}\int_0^\infty \mathrm{d}\alpha\, \alpha \left(f(\alpha) - (2\pi\alpha)^{-d/2}\right).$$

$a(d)$ depends on the regularization used for the lattice action

$$S(\vec{\phi}) = S_K(\vec{\phi}) + N \sum_x [V(\vec{\phi}^2)]$$

where

$$S_K(\vec{\phi}) = \tfrac{1}{2} N (2\pi)^d \int_{-\pi}^{\pi} \mathrm{d}^d k \, \tilde{\vec{\phi}}(-k) Q(k) \tilde{\vec{\phi}}(k), \quad \vec{\phi}(x) \equiv \int_{-\pi}^{\pi} \mathrm{d}^d k \ \mathrm{e}^{ikx} \, \tilde{\vec{\phi}}(k).$$

The lattice spacing is taken to be 1.

If only nearest-neighbor interactions are included:

$$Q(k) = 4 \sum_{\mu=1}^{d} \sin^2(\tfrac{k_\mu}{2}),$$

then

$$f(\alpha) = (\mathrm{e}^{-\alpha} \, I_0(\alpha))^d$$

where $I_0(\alpha)$ is a modified Bessel function.

In the following regularization which includes contributions from next to nearest neighbors and next to next to nearest neighbors the dimension d can be varied continuously:

$$S_K/N = -\tfrac{1}{2} \sum_x \phi(x) \Bigg[(1-r) \sum_{\mu=1}^{d} \big(\phi(x+\hat{\mu}) + \phi(x-\hat{\mu}) - 2\phi(x)\big) \\ + \tfrac{1}{4} r \sum_{\mu=1}^{d} \big(\phi(x+2\hat{\mu}) + \phi(x-2\hat{\mu}) - 2\phi(x)\big) \Bigg].$$

The inverse of the free field propagator is now:

$$Q(k;r) = \sum_{\mu=1}^{d} \left(4 \sin^2 \tfrac{k_\mu}{2} - 4r \sin^4 \tfrac{k_\mu}{2} \right).$$

Values of $a(d)$ in this regularization are exhibited in Fig. 2 below.

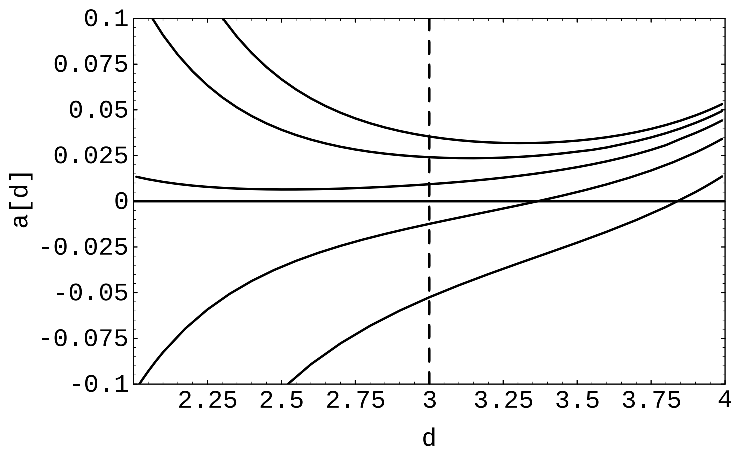

Fig. 2: Values of $a(d)$ in lattice regularization.
(From top to bottom: $r = -1.2,\ -0.8,\ -0.4,\ 0,\ 0.4.$)

At $d = 3$, including all next-to-nearest-neighbors we have:

$$S_K/N = -\tfrac{1}{2}\sum_x \phi(x)\left[(1-r)\sum_{\mu=1}^{3}\big(\phi(x+\hat\mu)+\phi(x-\hat\mu)-2\phi(x)\big)\right.$$
$$\left.+\tfrac{1}{4}r\sum_{3\geq\mu>\nu\geq1}\big(\phi(x+\hat\mu+\hat\nu)+\phi(x+\hat\mu-\hat\nu)+\phi(x-\hat\mu+\hat\nu)+\phi(x-\hat\mu-\hat\nu)-4\phi(x)\big)\right]$$

and thus

$$Q_2(k;r) = 4\sum_{\mu=1}^{3}\sin^2(\tfrac{k_\mu}{2}) - 2r\left[\left(\sum_{\mu=1}^{3}\sin^2(\tfrac{k_\mu}{2})\right)^2 - \sum_{\mu=1}^{3}\sin^4(\tfrac{k_\mu}{2})\right].$$

Once again, one finds that as the parameter r is growing, $a(d)$ is changing its sign from $a(d) > 0$ to $a(d) < 0$ (we have $a(d) = 0.002\ ,\ -0.0001\ ,\ -0.07$ for $r = -0.6\ ,\ -0.5\ ,\ 0.9$ respectively).

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